2) Both A and A_2 are singular (there are only zeros in the l th row or l th column), but A_1 formed from A_2 by striking out the l th row and l th column is nonsingular.

3) All three matrices A, A_2 and A_1 are singular.

In Case (1) we place zeros in the k th row and k th column of the matrix A', and the remaining elements are those of A_2^{-1}. In Case (2) the elements of the k th and l th rows and columns of the matrix A' are set equal to zero and in the remaining place we put A_1^{-1}. Finally, in Case (3) $A' \equiv 0$.

It is easy to see now that if the matrix Γ (4.1) is positive definite, then all the statements proved above for diagonal matrices A and B remain valid, since the quadratic forms $\gamma_{ij}^{(q)} u_i u_j$ entering into the relations (2.3) and (3.11) are also positive definite in the present case.

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ON THE ANALYSIS OF SHELLS WITH HOLES

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The possibility of application of Neumann's procedure to the investigation of shells with holes is examined.

The transfer of Neumann's method to shells is connected with two difficulties. First of all, the application of Kirchhoff's stresses does not reduce the problem directly to well-studied integral equations. Therefore, in this paper initially the investigation is related to the principal vector and moment. The latter circumstance, however, leads to singular integral equations which also have fixed singularities. However, the specific form of the resulting equations allows the establishment of Fredholm's alternative for these equations within the required limits. After the proof of Fredholm's alternative the convergence of Neumann's method is proven. The results of Fredholm present the possibility to establish convergence of Kirchhoff's stresses for sufficiently smooth contours of holes and load.

We shall examine shells with holes for which Neumann's method can be realized on electronic digital computers. It is shown that in the general case of loading this class of shells must have holes with a diameter of the order of $d \sim 4.5\ (Rh^2)^{1/3}$. This condition is compared with the known criterion of Lur'e [1], which was obtained for a cylindrical shell with a small circular hole.

Let the surface of the shell be bounded by the contour S_0 and let it have n stress-free holes with contours S_i and internal domains D_i. The inside of the shell is designated by D_0 . The domain contained within S_0 is called D. The domain which represents the union of all domains D_i is called D_1.

The displacement vector of the shell $\mathbf{w}$, in addition to satisfying the equations of equilibrium in D_0 and conditions of attachment on S_0 (which are not written out), must satisfy on S_i four relationships of the form

$$\mathbf{F} = 0, \qquad M = 0 \tag{1}$$

Condition (1) is represented in the following form:

$$\mathbf{F}^+ - \mathbf{F}^- = -\mathbf{F}^-, \qquad M^+ - M^- = -M^- \tag{2}$$

Here $\mathbf{F}$ is the principal vector, and M is the moment which acts on the internal contours of the holes [2] (p. 113). Conditions (1) guarantee the uniqueness of the problem which is being examined. The signs plus and minus indicate limiting values of quantities from inside and outside of contours S_i, respectively.

Let us introduce Green's tensor of the solid shell $G\ (x,\ y)$ and solve problem (2) by the following method:

$$\mathbf{u}_0 = \iint\limits_D G\ (x,\ y)\ \mathbf{f}\ (y)\ dy_1 dy_2, \ \ldots,\ \mathbf{u}_n = \sum_{i=1}^{4} \int\limits_{S_1} \mathbf{G}_i\ (x,\ s)\ T_{i\ (n-1)}^-\ (s)\ ds$$

$$\mathbf{w} = \mathbf{u}_0 + \mathbf{u}_1 + \ldots + \mathbf{u}_n + \ldots \tag{3}$$

Here $\mathbf{f}\ (x)$ is the surface loading, and S_1 is the contour which is the union of all S_i. Green's vectors $\mathbf{G}_i\ (x,\ y)$ on the contour have the following components:

$$G_1 = \omega_{tt}\ (G\ (x,\ y)),\quad G_3 = \theta_t\ (G\ (x,\ y)),\quad G_2 = -\varepsilon_{tt}\ (G\ (x,\ y)),\quad G_4 = \theta_\nu\ (G\ (x,\ y))$$

The quantities ω_{tt}, ε_{tt}, θ_t, θ_ν are the deformations [2] connected with the contour S_1. Here to obtain $\mathbf{G}_i\ (x,\ y)$, ω_{tt}, ε_{tt}, θ_t and θ_ν are fulfilled three times on each column $G\ (x,\ y)$ as a vector. In addition to this the following notation is introduced into (3)

$$T_1 = F_\nu,\quad T_2 = F_t,\quad T_3 = F_n,\quad T_4 = M$$

The following equation is applicable:

$$\mathbf{F} = \int\limits_{s_0}^{s} \mathbf{Q} ds \tag{4}$$

Here s_0 is some fixed point for each contour S_i.

From the determination of Green's tensor it follows [3] that if $\mathbf{f}\ (x) \in H\ (0,\ A,\ \lambda)$ and $S_1 \in Л_4\ (B,\ \lambda)$, then the solution according to method (3), if it converges, satisfies all necessary conditions for a shell with holes.

Here we accept results of Guenter [3]: vector $\mathbf{f}\ (x)$ belongs to class $H\ (n,\ A,\ \lambda)$ if for each two points x, x_0, the distance between which is $r\ (x,\ x_0)$, any of its components $f_i\ (x)$ has finite derivatives of the order n, which satisfy Hoelder's condition with constants A and λ

$$f_{in} = \frac{\partial^n f_i}{\partial x_1^p \partial x_2^q},\quad p + q = n,\quad |f_{in}(x)| \leqslant A$$

$$|f_{in}\ (x) - f_{in}\ (x_0)| \leqslant A r^\lambda\ (x, x_0)$$

The contour S belongs to the class $Л_n\ (B,\ \lambda)$, if the function which represents the equation of the contour in natural coordinates belongs to the class $H\ (n,\ B,\ \lambda)$. In these definitions the values A, B, λ do not depend on points x and x_0. We shall prove the convergence of (3) in Hoelder's norm. First of all let us agree to examine only that variant of the theory of shells in which the bending deformations are considered to be independent of the curvature [4] (p. 95). This condition allows the assumption that the principal part of Green's tensor has the same singularity as for plates

$$\Gamma_0(x, y) = (2\pi)^{-1}\|g_{kl}\| \qquad (k, l = 1, 2, 3) \tag{5}$$

$$g_{11} = n\ln r - m\left(\frac{\partial r}{\partial x_1}\right)^2, \quad g_{12} = -m\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2} \quad \left(n = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}\right)$$

$$g_{21} = -m\frac{\partial r}{\partial x_1}\frac{\partial r}{\partial x_2}, \quad g_{22} = n\ln r - m\left(\frac{\partial r}{\partial x_2}\right)^2 \quad \left(m = \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}\right)$$

$$g_{13} = g_{23} = g_{31} = g_{32} = 0, \qquad g_{33} = {}^1/_4\; r^2 \ln r$$

Here λ and μ are Lamé constants. Further, from equations (3) and (4) it follows that for any n:

$$\mathbf{F}_n(s_0) = 0 \tag{6}$$

We take into account that the principal singularities T_i are produced only by the higher derivatives of $\mathbf{w}$, having the following form [5, 6]

$$T'(\mathbf{w}) = \int_{s_0}^{s}\left(2\mu\frac{\partial \mathbf{w}_0}{\partial n} + \lambda\mathbf{n}\,\mathrm{div}\,\mathbf{w}_0 + \mu(\mathbf{n}\times\mathrm{rot}\,\mathbf{w}_0)\right)ds$$

$$T_3 = \int_{s_1}^{s}\left(\frac{\partial \Delta w_3}{\partial n} + (1-\sigma)\frac{\partial^3 w_3}{\partial n \partial s^2}\right)ds, \qquad T_4 = \sigma\Delta w_3 + (1-\sigma)\frac{\partial^2 w_3}{\partial n^2} \tag{7}$$

Here vectors $T'(\mathbf{w})$ and $\mathbf{w}_0$ are introduced which have two components each, T_1 and T_2 in w_1 and w_2, respectively. Now, on the basis of Eq. (3) and Eqs. (7), let us compute the vector $T^-\mathbf{w}_n$.

These calculations which are common in potential theory (for example [5], p. 268) lead to equations of the form

$$T^-\mathbf{w}_n = {}^1/_2\,T^-\mathbf{w}_{n-1} - \int_{S_1} TG(x, s)\,T^-\mathbf{w}_{n-1}(s)\,ds \tag{8}$$

The integral must be understood in the sense of a principal value of Cauchy. A peculiarity of Eqs. (8) will be that as a result of integration over the open contour from s_0 to s in (7) the singularity of the kernel $TG(s, s_1)$ has the form

$$K(s, s_1) = K_1(s, s_1) - K_1(s, s_0)$$

Here $K_1(s, s_1)$ represents a singular kernel of Cauchy, while $K_1(s, s_0)$ is a singular kernel of Cauchy with a fixed singularity s_0. From Eq. (8) we see that the applicability of method (3) is equivalent to the proof that Fredholm's theory is applicable to equation

$$\varphi(s) = \chi\left({}^1/_2\,\varphi(s) - \int_{S_1} TG(s, s_1)\,\varphi(s_1)\,ds_1\right) \tag{9}$$

and that its characteristic values are greater than unity. We rewrite system (9) in the form

$$\varphi(s) + \eta\int_{S_1} TG(s, s_1)\,\varphi(s_1)\,ds_1 = 0, \qquad \eta = \frac{2\chi}{2-\chi} \tag{10}$$

From Eqs. (7) we construct a symbolic matrix of system (10)

$$I_0 = \|p_{kl}\| \quad (k, l = 1, 2, 3, 4) \tag{11}$$

$$p_{31} = p_{41} = p_{32} = p_{42} = 0$$

$$p_{11} = p_{22} = p_{33} = p_{44} = 1, \quad p_{13} = p_{14} = p_{23} = p_{24} = 0 \qquad \text{(cont.)}$$
$$p_{12} = -p_{21} = {}^1\!/_4\eta i \ (1-\sigma), \ -p_{34} = p_{43} = {}^1\!/_4 \ \eta i \ (1+\sigma)$$

Here σ is Poisson's ratio.

In the construction of the symbolic matrix the fixed singularity was not taken into consideration. The values of χ corresponding to det $I_0 = 0$ for all physically conceivable values of σ do not fall into the interval $-1 \leqslant \chi \leqslant 1$. Furthermore, the operator

$$\mathbf{R} = I\varphi(t) - \eta K \left[\int_{S_1} \frac{\varphi(z)\,dz}{t-z} - \int_{S_1} \frac{\varphi(z)\,dz}{s_0 - z} \right]$$

in which I is a unit matrix, K is a constant matrix which is related to I_0 in a known manner, regularizes (10) for values of χ when det $I_0 \neq 0$, i.e. it reduces (10) to Fredholm's equation.

Now we prove that equation $\mathbf{R} = 0$ has only trivial zeros for values of χ, which satisfy the condition of regularization. Direct substitution shows that the constant vector cannot be a zero of the equation $\mathbf{R} = 0$ and that all zeros of the equation

$$\mathbf{R}_1 = I\varphi(t) - \eta K \int_{S_1} \frac{\varphi(z)\,dz}{t-z} \tag{12}$$

differ by a constant vector from zeros of the equation $\mathbf{R} = 0$ and vice versa. However, Eq. (12) for χ which satisfy the condition of regularization has only trivial zeros ([7], Sect. 63). This means that the same can be said about zeros of the equation $\mathbf{R} = 0$. Thus, for Eq. (9) Fredholm's theory holds. The proof that the characteristic values are real and not confined to the interval $-1 < \chi < 1$ proceeds in the classical manner [8]. For this purpose we note that having constructed the potential

$$\mathbf{V} = \mathbf{V}_1 + i\mathbf{V}_2 = \sum_{i=1}^{4} \int_{S_1} \mathbf{G}_i(x, s)\,\varphi_i(s)\,ds, \qquad \eta = \eta_1 + i\eta_2$$

from Eq. (10) we obtain $\varphi(s_0) = 0$ and according to equations of jumps

$$\varphi(s) = T^+\mathbf{V} - T^-\mathbf{V}$$

$$\sum_{i=1}^{4} \int_{S_1} T\mathbf{G}_i(s, s_1)\,\varphi_i(s_1)\,ds_1 = \frac{T^+\mathbf{V} + T^-\mathbf{V}}{2} \tag{13}$$

Substituting this into (10), separating the real and imaginary parts, multiplying the scalar real part by the vector $(\omega_{tt}, -\varepsilon_{jj}, \theta_t, \theta_\nu)\,\mathbf{V}_1$, and the imaginary part by the vector $(\omega_{tt}, -\varepsilon_{tt}, \theta_t, \theta_\nu)\mathbf{V}_2$, and subtracting one from the other we obtain after integration over S_1 with consideration of conditions on S_0 the following expression:

$$(2-\eta_1)\ (W_0 + W_1) + (2+\eta_1)\ (W'_0 + W_1') = 0 \tag{14}$$

Multiplying the real part of (10) by $(\omega_{tt}, -\varepsilon_{tt}, \theta_t, \theta_\nu)\,\mathbf{V}_2$ and the imaginary part by $(\omega_{tt}, -\varepsilon_{tt}, \theta_t, \theta_\nu)\,\mathbf{V}_1$, after addition we integrate the obtained sum over S_1, taking into account conditions on S_0. As a result we find

$$\eta_2\,(W_0 + W_1) - \eta_2\,(W_0' + W_1') = 0 \tag{15}$$

In the last two equations W_0 and W_1, respectively, are the potential energy of deformation corresponding to the displacement $\mathbf{V}_1$ of solid domains D_0 and D_1. W_0' and W_1' are the potential energies of deformation of the same domains corresponding to vector $\mathbf{V}_2$.

As a result of positiveness of energy it follows from (14) and (15) that $\eta_2 = 0$, $|\eta_1| \geqslant 2$. The proof that $\eta_1 = 2$ will not be a characteristic value of (10) is based on the uniqueness of the internal problem of Dirichlet and is carried out as in [5] (p. 345). Everything presented so far leads to justification of method (3) in Hoelder's norm. After the convergence has been proven there is no necessity in (3) to compute each time the principal vector and moment. By virtue of uniqueness theorems we can compute the usual Kirchhoff conditions on the contour Q (w) and the moment M.

This is also related to the fact that in the case $\mathbf{f} \in H(0, A, \lambda)$ and $S_1 \in Л_4(B, \lambda)$ series (8) allow differentiation. From the representation of Fredholm's resolvent in the form of a quotient of entire functions it follows that the derivatives converge under the same conditions as F. Therefore, instead of (8) we can compute at each stage

$$Q^-(\mathbf{w}_n) = \tfrac{1}{2} Q^-(\mathbf{w}_{n-1}) - \sum_{i=1}^{4} \int_S Q\,[\mathbf{G}_i'(x, s)\, Q_i^-(\mathbf{w}_{n-1})]\, ds \qquad (16)$$

Here vectors $\mathbf{G}_i'(x, s)$ are displacements of the shell due to "unit" forces applied to S_1; the quantities Q_i $(i = 1, 2, 3)$ are components of Q (**w**) and $Q_4 = M$.

It should be emphasized that method (3) represents a transfer of Neumann's method of solution of a boundary value problem with given normal derivative for the Laplace equation [9] to shells. It is also necessary to note that the numerical construction of the solution according to method (3) at the present time is connected with considerable difficulties [10].

Let us examine the conditions of application of method (3) with present computation techniques. The principal difficulty of computation [10] is connected with the presence of terms of the form w / R in the expressions for the elongations ε_{ij}, and also with the earlier mentioned fact that the principal part of Green's tensor in the theory of shells is the same as for plates. In the implementation of method (3) therefore terms of the following type appear:

$$\frac{\alpha}{h} + \frac{\beta l}{Rh^3} \qquad (17)$$

As a consequence of the small thickness h in the general case of loading of an arbitrary shell the quantity β must be computed with accuracy to the order of h^2, which apparently cannot be realized with existing electronic digital computers. Therefore, we select a class of shells for which both terms in expression (17) are of the same order. From expressions (3) and (5) and equations for boundary stresses [4] we can estimate the norm in L_2 of normal stresses on the contour S_1 in any approximation. Then in expression (17) we shall have instead of the first term $\alpha_1 / hd^{1/2}$, and instead of the second term $\alpha_2 d^{5/2}/2\pi Rh^3$. The quantities α_1 and α_2 are approximately the same, R is the minimum radius of curvature, and d is the characteristic diameter of the hole or the distance between the holes. If it is required that the last expression exceed the first (to make numerical computation possible) by no more than 15-20 times, an expression is obtained for the maximum d

$$d \approx 4.5\ (Rh^2)^{1/3} \qquad (18)$$

We note that condition (18) for commonly applied shells numerically almost coincides with the condition $d^2 \ll 4\ Rh$, which was obtained in [1] for a cylindrical shell. It should be stated that fulfilment of condition (18) guarantees the possibility of calculation for a shell with arbitrary holes and with arbitrary loading at the same time when the solution of Lur'e is applicable only to an initially moment-free cylindrical shell with one small circular hole.

Further strengthening of condition (18) is connected with clarification of the form of the state of stress of the shell. Thus, $\beta = 0$ in cases of pure moment or moment-free stressed conditions. Therefore, in spite of the fact that the method of Neumann (3) strictly mathematically always converges, its numerical application with present computational possibilities in the general case of loading is limited by condition (18). Therefore for $R = (100\text{-}1000)h$ we shall have $d \leqslant (20\text{-}40)h$. This class of shells presents a practical interest.

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ON A MODEL OF A MEDIUM WITH COMPLEX STRUCTURE

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In recent years the attention of a large number of investigators has been drawn to the study of media having complex structure. The simplest of these is the Cosserat medium [1, 2]. Mindlin's medium with microstructure [3] is more complex. An extraordinary complexity is inherent in the multipolar mechanics developed by Green and Rivlin [4].

The essential peculiarity of all these theories is reconsideration of the concept of a point. If in classical continuum mechanics each point possesses only the degrees of freedom of translational displacement, in the Cosserat theory the degrees of freedom of a rigid body are ascribed to it. In the theory of a medium with microstructure each point possesses the degrees of freedom of a body with homogeneous strain, i. e. twelve degrees of freedom. In multipolar mechanics the mechanical state of each point is defined by n kinematical parameters, where n must be finite but may be as large as desired. A new model of a medium of similar type will be constructed below.

We shall postulate the presence of some load-carrying medium and shall assume that